

Integrals of general birth-death processes

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Abstract

Integral functionals of Markov processes are widely used in stochastic modeling for applications in ecology, evolution, infectious disease epidemiology, and operations research. The integral of a stochastic process is often called the “cost” or “reward” accrued by the process. Many important stochastic counting models can be written as general birth-death processes (BDPs), which are continuous-time Markov chains on the non-negative integers in which only jumps to adjacent states are allowed, and there are no jumps down from zero. While there has been considerable progress in understanding general BDPs, work on integral functionals of BDPs has been limited to simple models and moments. In this paper, we show how to compute the distribution function and probability density of integral functionals for any general BDP. The method allows routine evaluation of full probability distributions for prospective modeling in several popular biological models and provides a framework for computing likelihoods that may be useful for statistical inference in a variety of applied scenarios. We provide examples of previously intractable integrals of BDPs from biology, queuing theory, population genetics, finance, and infectious disease epidemiology. In the final example, we use concepts from operations research to show how to select a control parameter to obtain exact probabilistic bounds on the total cost of an epidemic.

Keywords: General birth-death process, path integral, first passage time, total cost, epidemic

1 Introduction

Many important real-life applications of stochastic models can be characterized as questions about path integrals of Markov processes. For example, the total cost of an infectious disease epidemic is the area under the time trajectory of the number of infected people (Jerwood, 1970; Gani and Jerwood, 1972). In ecology, the total time lived by a members of a population of organisms might indicate the success or failure of conservation efforts (Faith, 1992; Crawford and Suchard, 2012a). In finance, an investor earns dividends in proportion to the performance of a security over time. In operations research, managers desire the quantity of output produced or resource consumed over time. One reason for the practical usefulness of integral summaries of stochastic process realizations is revealed by the units of the integral, which is often expressed as (objects) \times (time). For example, in queuing theory, the time integral of the number of customers in a queue might be expressed in units of *person-hours*; in cancer research, the time integral of the number of animals alive during a trial of a radiation therapy technique might be expressed as *monkey-years* (Broerse et al, 1981); traffic engineers may be interested in the number of *vehicle-hours* waited in models for highway accident delays (Gaver, 1969).

A wide variety of discrete-valued stochastic models can be written as birth-death processes (BDPs), a class of continuous-time Markov chains taking values on the non-negative integers \mathbb{N} in which only jumps to adjacent states are allowed, and there are no jumps down from zero (Kendall, 1948). In applications, researchers usually treat a general BDP as a model for counting the number of hypothetical “particles”

(objects, organisms, species, infected people, etc.) in a system over time, where the particles can give birth and die. When a general BDP $X(t)$ is in state n , a “birth” (jump to $n + 1$) happens with instantaneous rate λ_n and a “death” (jump to $n - 1$ for $n > 0$) happens with instantaneous rate μ_n . The study of general BDPs enjoyed wide interest following the groundbreaking work of Karlin and McGregor (1957a,b, 1958), who derive expressions for transition probabilities, moments, and first passage times in terms of orthogonal polynomials. In the classical simple linear BDP, also known as the Kendall process, the per-particle birth and death rates are constant: $\lambda_n = n\lambda$ and $\mu_n = n\mu$ (Kendall, 1949). In a general BDP, λ_n and μ_n are arbitrary functions of n , but are assumed to be time-homogeneous. In practice, the orthogonal polynomials and spectral measure introduced by Karlin and McGregor (1957a,b) can be extremely difficult to derive for general BDPs (Novozhilov et al, 2006; Renshaw, 2011).

In this paper, we focus on integral functionals of BDPs. Let $g : \mathbb{N} \rightarrow [0, \infty)$ be an arbitrary positive function and let S be a set of “taboo” or prohibited states. Suppose the initial state of the BDP is $X(0) = i \in \mathbb{N} \setminus S$. This paper deals with the distribution of the integral functional

$$W_i = \int_0^{\tau_i} g(X(t)) dt. \quad (1)$$

where the upper limit of integration is the first passage time

$$\tau_i = \inf \{t : X(t) \in S \mid X(0) = i\}. \quad (2)$$

Here, W_i is a functional because it maps a realization of the stochastic process $g(X(t))$ to its integral. Figure 1 shows an example realization of a BDP and its integral W_i with $S = \{0\}$. The left-hand side shows a BDP beginning at $X(0) = 1$, and ending at $X(\tau_1) = 0$. The right-hand plot shows $g(X(t))$ over the same time interval, and the area under the trajectory is W_i .

The work of Karlin and McGregor (1957a,b) provided the first theoretical tools for working with integral functionals of general BDPs. Puri (1966, 1968) derives the characteristic function for the joint distribution of simple linear BDP and its integral and gives expressions for moments and limiting distributions. In a series of three subsequent papers, Puri further develops the theory of integrals of general stochastic processes and certain BDPs (Puri, 1971, 1972a,b). McNeil (1970) gives the first results for general BDPs, Gani and McNeil (1971) derive expressions for the joint distribution of a general BDP and its integral, and Kaplan (1974) provides limit theorems for integrals of simple BDPs with immigration. Ball and Stefanov (2001) interpret BDPs as exponential families to derive generating functions for the distribution of first passage times and moments of stopped reward functions. More recently, researchers have found straightforward methods for finding moments of integrals of general BDPs using Laplace transforms (Hernández-Suárez and Castillo-Chavez, 1999; Pollett and Stefanov, 2003; Pollett, 2003; Gani and Swift, 2008). Alongside these advances related to integrals of BDPs, several researchers have made progress in characterizing a wide variety of interesting quantities related to BDPs in terms of continued fractions (Murphy and O’Donohoe, 1975; Jones and Magnus, 1977; Bordes and Roehner, 1983; Guillemin and Pinchon, 1998, 1999). Flajolet and Guillemin (2000) introduce a combinatorial interpretation and more continued fraction expressions for interesting quantities related to BDPs, and Crawford and Suchard (2012b) show how to compute transition probabilities for general BDPs using numerical inversion of continued fraction expressions for Laplace transforms.

Integrals of functionals of BDPs have garnered extensive attention for their usefulness in applications as well. Epidemiologists have a special interest in integrals like (1); they call this the “total cost” of an epidemic (Jerwood, 1970; Gani and Jerwood, 1972; Downton, 1972; Ball, 1986). Gani and McNeil (1971) discusses applications to parking lot occupancy, Gaver (1969) discusses highway traffic delays, and Pace and Khaluf (2012) study applications to swarm robotics. In the operations research literature, Lefèvre (1981) discusses optimal control of epidemics that can be expressed as general BDPs; reward/cost models are often called “Markov decision processes” in this field. However, most analyses of integral functionals of general BDPs are limited to simple analytically tractable models or focused on moments of integrals like (1).

In this paper, we provide the first method for evaluating the distribution of (1) for an arbitrary general BDP. We first outline the basic theory of general birth-death processes and provide expressions for the Laplace transform of the transition probability, first passage time distributions, and distributions of integral functionals of general birth-death processes. Most previous work has focused on simple, analytically

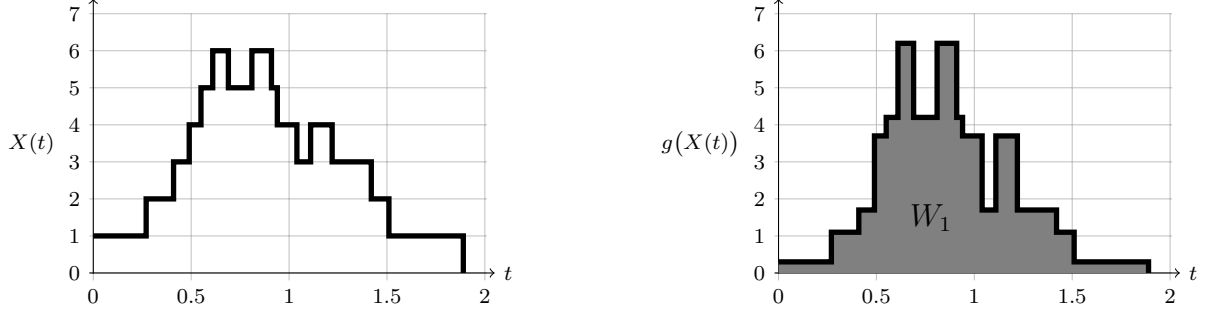


Figure 1: Illustration of the integral of a functional of a general birth-death process (BDP). On the left, a BDP begins at $X(0) = 1$ and ends when the process reaches the absorbing state 0 just before time $t = 2$. On the right, $W_1 = \int_0^{\tau_1} g(X(t)) dt$ is the area under the trajectory of $g(X(t))$, where $g : \mathbb{N} \rightarrow [0, \infty)$ is an arbitrary positive “reward” or “cost” function. The upper limit of integration τ_1 is the first passage time to zero, beginning at $X(0) = 1$. This paper is concerned with finding the probability distribution of quantities like W_1 over all realizations of $X(t)$. In this example, we have used a reward function $g(n)$ that is increasing in n , but this need not be the case in general.

tractable models, systems at equilibrium, or moments, which often have closed-form solutions (see, e.g. Hernández-Suárez and Castillo-Chavez (1999)), but often do not capture the variability in outcomes that might have initially motivated researchers to construct the stochastic models in the first place. In contrast, we establish explicit expressions for the Laplace transform of the relevant integrals using continued fractions and show how to evaluate these and invert them in a computationally efficient manner. The method does not require knowledge of the orthogonal polynomials and spectral measure introduced by Karlin and McGregor (1957a,b). These developments make possible routine computation of probability distributions of integral functionals of general birth-death processes, and have the potential for wide use as predictive models in biology, medicine, and finance. The paper therefore concludes with an extensive set of concrete examples in which we demonstrate the distribution of certain integral functionals of previously intractable BDP models.

2 Mathematical theory

Let $X(t)$ be a general BDP with transition rates λ_k and μ_k , $k = 0, 1, \dots$ with $\mu_0 = 0$. Then the transition probabilities $P_{ij}(t) = \Pr(X(t) = j \mid X(0) = i)$ satisfy the forward system of ordinary differential equations

$$\begin{aligned} \frac{dP_{i0}(t)}{dt} &= \mu_1 P_{i1}(t) - \lambda_0 P_{i0}(t), \text{ and} \\ \frac{dP_{ij}(t)}{dt} &= \lambda_{j-1} P_{i,j-1}(t) + \mu_{j+1} P_{i,j+1}(t) - (\lambda_j + \mu_j) P_{ij}(t), \end{aligned} \quad (3)$$

with $j \geq 1$ with $P_{ii}(0) = 1$ and $P_{ij}(0) = 0$ for $i \neq j$ (Feller, 1971). The corresponding backward equations are

$$\frac{dP_{ij}(t)}{dt} = \lambda_j P_{i,j-1}(t) + \mu_j P_{i,j+1}(t) - (\lambda_j + \mu_j) P_{ij}(t). \quad (4)$$

2.1 Transition probabilities

To solve for the transition probabilities, it is advantageous to work in the Laplace domain (Karlin and McGregor, 1957b). The presentation is based on that of Crawford et al (2011). Let

$$f_{ij}(s) = \mathbb{E} [e^{-st}] = \int_0^\infty e^{-st} P_{ij}(t) dt \quad (5)$$

be the Laplace transform and $\delta_{ij} = 1$ if $i = j$ and zero otherwise. Laplace transforming equation (3) yields

$$\begin{aligned} (s + \lambda_0)f_{i0}(s) - \delta_{i0} &= \mu_1 f_{i1}(s) \\ (s + \lambda_j + \mu_j)f_{ij}(s) - \delta_{ij} &= \lambda_{j-1}f_{i,j-1}(s) + \mu_{j+1}f_{i,j+1}(s) \end{aligned} \quad (6)$$

after straightforward rearrangement of terms. Letting $i = 0$, we find that

$$\begin{aligned} f_{00}(s) &= \frac{1}{s + \lambda_0 - \mu_1 \frac{f_{01}(s)}{f_{00}(s)}}, \text{ and} \\ \frac{f_{0j}(s)}{f_{0,j-1}(s)} &= \frac{\lambda_{j-1}}{s + \mu_j + \lambda_j - \mu_{j+1} \frac{f_{0,j+1}(s)}{f_{0j}(s)}}. \end{aligned} \quad (7)$$

Solving for $f_{00}(s)$, we arrive at the continued fraction expression

$$f_{00}(s) = \frac{1}{s + \lambda_0 - \frac{\lambda_0 \mu_1}{s + \lambda_1 + \mu_1 - \frac{\lambda_1 \mu_2}{s + \lambda_2 + \mu_2 - \dots}}}. \quad (8)$$

Letting $a_1 = 1$, $a_j = -\lambda_{j-2}\mu_{j-1}$, $b_1 = s + \lambda_0$, and $b_j = s + \lambda_{j-1} + \mu_{j-1}$ for $j \geq 2$, (8) becomes

$$f_{00}(s) = \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \dots}}} = \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} \dots, \quad (9)$$

in more succinct notation. Define the k th convergent of $f_{00}(s)$ as the rational function

$$f_{00}^{(k)}(s) = \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \dots \frac{a_k}{b_k} = \frac{A_k(s)}{B_k(s)}. \quad (10)$$

With these expressions in hand, we give a fundamental result on transition probabilities for general BDPs.

Theorem 1. *The Laplace transform of the transition probability $P_{ij}(t)$ in a general BDP is*

$$f_{ij}(s) = \begin{cases} \left(\prod_{k=j+1}^i \mu_k \right) \frac{B_j(s)}{B_{i+1}(s)+} \frac{B_i(s)a_{i+2}}{b_{i+2}+} \frac{a_{i+3}}{b_{i+3}+} \dots & \text{for } j \leq i, \\ \left(\prod_{k=i}^{j-1} \lambda_k \right) \frac{B_i(s)}{B_{j+1}(s)+} \frac{B_j(s)a_{j+2}}{b_{j+2}+} \frac{a_{j+3}}{b_{j+3}+} \dots & \text{for } i \leq j. \end{cases} \quad (11)$$

The proof of this Theorem is given in Crawford and Suchard (2012b).

2.2 Computational inversion

With explicit expressions for the Laplace transform of the transition probability for any general BDP in hand, we now turn to the task of computational inversion of (11). Three advantageous facts make numerical inversion possible. First, there exist stable and efficient recursive schemes for evaluating continued fractions to an arbitrary depth; the pioneering work by Wallis (1695) and more recent developments by Lentz (1976), Thompson and Barnett (1986) and popularization by Press (2007) allow routine evaluation of continued fractions. Second, there are good *a posteriori* bounds for truncation of continued fractions of this type which allow evaluation of the error due to stopping the evaluation at a finite depth (Craviotto et al, 1993). Finally, numerical Laplace inversion for well-behaved probability densities is straightforward. We appeal

to the methods popularized by Abate and Whitt (1992a) and Abate and Whitt (1992b). Our approach involves numerical evaluation of the continued fraction and numerical Laplace inversion while controlling the error due to truncation of the continued fraction and discretization of the inversion integral. The method is described in greater detail in Crawford and Suchard (2012b), but we briefly outline the main concepts.

Following Abate and Whitt (1992a) and Crawford and Suchard (2012b), we approximate the integral above by a discrete Riemann sum as follows:

$$P_{ij}(t) = \frac{e^{A/2}}{2t} \operatorname{Re} \left(f_{ij} \left(\frac{A}{2t} \right) \right) + \frac{e^{A/2}}{t} \sum_{k=1}^{\infty} (-1)^k \operatorname{Re} \left(f_{ij} \left(\frac{A + 2k\pi\sqrt{-1}}{2t} \right) \right). \quad (12)$$

where $\operatorname{Re}(s)$ is the real part of the complex variable s . The error due to discretization in (12) is

$$e_d = \sum_{k=1}^{\infty} e^{-kA} P_{ij}((2k+1)t) \leq \sum_{k=1}^{\infty} e^{-kA} = \frac{e^{-A}}{1 - e^{-A}} \approx e^{-A}, \quad (13)$$

where the second relation follows when $P_{ij}(t) \leq 1$, and the last when e^{-A} is small (Abate and Whitt, 1992a). To obtain $e_d \leq 10^{-\gamma}$, we set $A = \gamma \log(10)$. This establishes the error due to discretization of the Bromwich integral for Laplace inversion. However, the expressions (11) are infinite continued fractions which can only be evaluated to finite depth in practice. We therefore seek a bound on the error due to truncation of the continued fraction Laplace transform. Craviotto et al (1993) demonstrate the *a posteriori* error bound for convergents of continued fractions of the form (8):

$$\left| f_{00}(s) - f_{00}^{(k)}(s) \right| \leq \frac{\left| \frac{B_k(s)}{B_{k-1}(s)} \right|}{\left| \operatorname{Im} \left(\frac{B_k(s)}{B_{k-1}(s)} \right) \right|} \left| f_{0,0}^{(k)}(s) - f_{0,0}^{(k-1)}(s) \right|, \quad (14)$$

whenever $\operatorname{Im}(s)$, the imaginary part of the complex variable s , is nonzero. The expression on the right-hand side of (14) is easy to evaluate in general since one always finds $f_{00}^{(k-1)}(s)$ before computing $f_{00}^{(k)}(s)$. In this way, we can control the error due to discretization in (12) and truncation in (14). Finally, we note that both (11) and (14) require evaluation of ratios of denominators of continued fraction convergents. Computation of these ratios can be highly unstable using standard methods for computing convergents; Appendix A shows how to evaluate this ratio in a numerically stable way.

2.3 First passage times

Now consider the time of first arrival of a BDP $X(t)$ into an arbitrary set S of taboo states, and suppose $X(0) = i \in \mathbb{N} \setminus S$. This first passage time is defined formally as

$$\tau_i = \inf \{t : X(t) \in S \mid X(0) = i\}. \quad (15)$$

To find the relationship between first passage times and the expressions for transition probabilities discussed above, construct a new process $Y(t)$ identical to $X(t)$ except that $\lambda_j = \mu_j = 0$ for every $j \in S$, so every state in S is absorbing. Then for this modified process, with $P_{ij}(t) = \Pr(Y(t) = j \mid Y(0) = i)$,

$$\Pr(\tau_i < t) = \sum_{j \in S} P_{ij}(t). \quad (16)$$

The intuitive reason for this equality is the absorbing nature of the states in S : if Y reaches a state $j \in S$ at any time before t , we must have $Y(t) = j$. Furthermore, Y cannot visit more than one state in S , so the absorption events are mutually exclusive and the probability of absorption is simply the sum of the individual absorption probabilities. Therefore the cumulative distribution function of the first passage time into S is given by the sum of the transition probabilities from i to every taboo state in S for the modified process $Y(t)$. This observation provides the necessary link between first passage time distributions and transition probabilities that we will need in order to attack integrals like (1).

Before we proceed with our discussion of integrals of BDPs, we must account for the possibility that the process first passage time may not always exist (Karlin and McGregor, 1957b). In other words, the birth

and death rates for a general BDP may be such that the process “runs away” to infinity in finite time. This is also known as explosive growth. Formally, suppose the process begins at $X(0) = 0$ and there are no absorbing states. Renshaw (2011) shows that the expected first passage time to infinity τ^∞ is

$$\mathbb{E}(\tau^\infty \mid X(0) = 0) = \sum_{j=0}^{\infty} \pi_j \sum_{n=j}^{\infty} \frac{1}{\mu_n \pi_n} \quad (17)$$

where $\pi_1 = 1$ and

$$\pi_n = \prod_{k=1}^n \frac{\lambda_{k-1}}{\mu_k} \quad (18)$$

for $i > 1$. When (17) diverges, the process is non-explosive, and the first passage time from 0 to any finite state j is almost surely finite. When (17) is finite, the first passage time to infinity is infinite with non-zero probability. We return to this topic, in the context of integral functionals, in the next section.

2.4 Integral functionals of BDPs

Now we consider the problem of computing the distribution of (1). Our emphasis on first-passage times as the upper limit of integration in (1) has two benefits. First, our analyses need not be conditional on an arbitrary time in the future. Second, first passage times allow us to exploit powerful analytic tools that establish a correspondence between transition probabilities and first-passage times, allowing us to make analytic progress on integrals for arbitrary well-behaved processes. Our presentation follows the outline given by McNeil (1970). Let

$$w_i(s) = \mathbb{E} [e^{-sW_i}] \quad (19)$$

be the Laplace transform of W_i . Recall that the destination state and waiting time U until the first departure from a given state are independent in a continuous-time Markov process, so the joint probability factorizes as follows:

$$\begin{aligned} \Pr(\text{birth}, U = u \mid X(0) = i) &= \Pr(\text{birth} \mid X(0) = i) \times \Pr(U = u \mid X(0) = i) \\ &= \frac{\lambda_i}{\lambda_i + \mu_i} \times (\lambda_i + \mu_i) e^{-(\lambda_i + \mu_i)u} \\ &= \lambda_i e^{-(\lambda_i + \mu_i)u}. \end{aligned} \quad (20)$$

Likewise, the probability of a death at time u is

$$\Pr(\text{death}, U = u \mid X(0) = i) = \mu_i e^{-(\lambda_i + \mu_i)u}. \quad (21)$$

If the process spends time u in state i before jumping to another state, then the accumulated reward/cost is $ug(i)$. Note that if $X(0) = i \in S$ then $\tau_i = 0$, $W_i = 0$, and so $w_i(s) = 1$. Now by an analogous conditioning argument for $X(0) = i \notin S$, we re-write (19) as

$$\begin{aligned} w_i(s) &= \int_0^\infty \mathbb{E} [e^{-s(W_{i+1} + ug(i))}] \Pr(\text{birth}, U = u \mid X(0) = i) \, du \\ &\quad + \int_0^\infty \mathbb{E} [e^{-s(W_{i-1} + ug(i))}] \Pr(\text{death}, U = u \mid X(0) = i) \, du \\ &= \mathbb{E} [e^{-sW_{i+1}}] \int_0^\infty e^{-sug(i)} \lambda_i e^{-(\lambda_i + \mu_i)u} \, du \\ &\quad + \mathbb{E} [e^{-sW_{i-1}}] \int_0^\infty e^{-sug(i)} \mu_i e^{-(\lambda_i + \mu_i)u} \, du \\ &= w_{i+1}(s) \lambda_i \int_0^\infty e^{-u(sg(i) + \lambda_i + \mu_i)} \, du \\ &\quad + w_{i-1}(s) \mu_i \int_0^\infty e^{-u(sg(i) + \lambda_i + \mu_i)} \, du \end{aligned} \quad (22)$$

which gives

$$(sg(i) + \lambda_i + \mu_i)w_i(s) = \lambda_i w_{i+1}(s) + \mu_i w_{i-1}(s). \quad (23)$$

Now dividing both sides of the above by $g(i)$, we find that

$$(s + \lambda_i^* + \mu_i^*)w_i(s) = \lambda_i^* w_{i+1}(s) + \mu_i^* w_{i-1}(s) \quad (24)$$

where $\lambda_i^* = \lambda_i/g(i)$ and $\mu_i^* = \mu_i/g(i)$. Therefore, we see that (24) is simply the backward equation for a modified process with birth and death rates λ_i^* and μ_i^* for $i \in \mathbb{N}$. The forward equation for the cumulative distribution function of W_i is therefore equivalent to (3) with the modified birth and death rates.

This remarkable fact has a simple heuristic explanation (Pollett and Stefanov, 2003). Note that we can decompose W_i as random sum of independent and identically distributed exponential variables

$$W_i = \sum_{n=0}^{\infty} g(n)T_n = \sum_{n=0}^{\infty} g(n) \sum_{k=0}^{N_n} T_{nk} \quad (25)$$

where T_n is the total time spent (sojourn time) in state n , N_n is the number of visits to state n , and T_{nk} is Exponential($\lambda_n + \mu_n$) waiting time during the k th visit to state n . Multiplying each T_{nk} by $g(n)$, we find that

$$W_i = \sum_{n=0}^{\infty} \sum_{k=0}^{N_n} \text{Exponential}(\lambda_n^* + \mu_n^*), \quad (26)$$

This informal derivation demonstrates that the reward/cost accumulated until the first passage time to a taboo state is equivalent to a first passage time under a process with modified transition rates. Pollett (2003) gives the conditions, analogous to those for (17), under which this modified process explodes. We note that differentiation of solutions of (24) yields moments of W_i , as noted by McNeil (1970) and subsequently refined by Hernández-Suárez and Castillo-Chavez (1999), Stefanov and Wang (2000), and Pollett (2003). We refer interested readers to those papers and focus here on results for the distribution of W_i , which are more useful in statistical and decision applications.

To take advantage of (24), we modify (15) as follows. Fix $S \subset \mathbb{N}$ and suppose $X(t)$ is a general BDP with rates $\{\lambda_n\}$ and $\{\mu_n\}$ with starting state $X(0) = i \in \mathbb{N} \setminus S$. Suppose $g(n)$ is a positive function defined for all $n \in \mathbb{N}$. Let $Y(w)$ be a general BDP with rates $\lambda_n^* = \lambda_n/g(n)$ and $\mu_n^* = \mu_n/g(n)$ for all $n \in \mathbb{N} \setminus S$, and $\lambda_n^* = \mu_n^* = 0$ for every $n \in S$. Then let $P_{ij}^*(t) = \Pr(Y(t) = j \mid Y(0) = i)$. We then have

$$\Pr(W_i < w) = \sum_{j \in S} P_{ij}^*(w). \quad (27)$$

If instead of the cumulative distribution function of W_i we wish to have the probability density, we could numerically differentiate (27). However, using the properties of the Laplace transform,

$$\begin{aligned} h(w) &= \frac{d}{dw} \Pr(W_i < w) \\ &= \sum_{j \in S} \frac{d}{dw} P_{ij}^*(w) \\ &= \sum_{j \in S} \mathcal{L}^{-1} \left[s f_{ij}^*(s) - P_{ij}^*(0) \right] (w) \\ &= \sum_{j \in S} \mathcal{L}^{-1} \left[s f_{ij}^*(s) \right] (w) \end{aligned} \quad (28)$$

where $f_{ij}^*(s)$ is the Laplace transform of $P_{ij}^*(t)$, $\mathcal{L}^{-1}[\cdot]$ denotes Laplace inversion, and $P_{ij}^*(0) = 0$ for all $j \in S$ since we have assumed $i \notin S$.

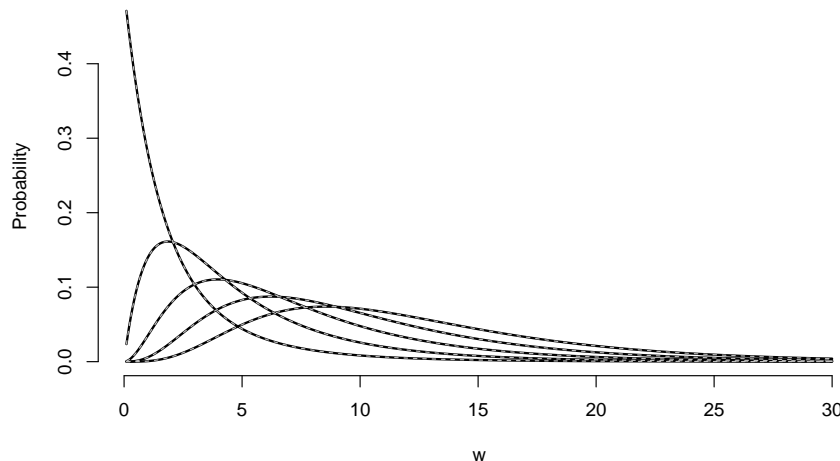


Figure 2: Probability density of the integral of a Kendall process with $\lambda = 0.1$, $\mu = 0.5$, starting at $X(0) = 1, 2, 3, 4, 5$ until extinction. The thick black traces are exact solutions from (29) and the red traces are the numerical solutions obtained using the method outlined in section 2.4. The numerical and exact solutions agree to high accuracy, but the numerical method works for any general BDP and does not require analytic derivation of orthogonal polynomials for the process.

3 Applications

In this section, we discuss several practical applications of our method. The first example illustrates that our numerical procedure produces results that agree with known analytic expressions. We then present several models for which there do not exist tractable solutions. In each case, we specify the birth and death rates $\{\lambda_n\}$ and $\{\mu_n\}$ for $n = 0, 1, 2, \dots$, the set of taboo states S , and the reward/cost function $g(n)$. In every example, we are able to generalize previous results to obtain exact distribution functions and probability densities for a variety of interesting integral quantities. In the final example, we present a model for the spread of an infectious disease and show how to set a control parameter to obtain a guaranteed probabilistic bound on its total cost.

3.1 Kendall process

In a simple linear BDP, also known as a Kendall process (Kendall, 1949), $\lambda_n = n\lambda$ and $\mu_n = n\mu$. Note that $\lambda_0 = 0$ so $S = \{0\}$ is absorbing and corresponds to extinction in ecological and evolutionary models. The Kendall process is widely used in ecology as a model for reproducing and dying organisms, evolution, genetics, and epidemiology. To analyze integrals of the Kendall process, let $g(n) = n$ for all n , so W_i is the total time lived by all particles/organisms until extinction. Under this model, $\lambda_n^* = \lambda$ for $n \geq 1$, $\lambda_0^* = 0$, and $\mu_n^* = \mu$ with $\mu_0^* = 0$, which is simply the $M/M/1$ queue, also known as the immigration-emigration model. McNeil (1970) and Pollett and Stefanov (2003) give an explicit expression for the probability density of W_i for $\mu > \lambda$ (since τ_i is almost surely finite for all i):

$$d\Pr(W_i \leq w) = \frac{i}{w} e^{-(\lambda+\mu)w} (\mu/\lambda)^{i/2} I_i(2w\sqrt{\lambda\mu}) dw \quad (29)$$

where $I_i(x)$ is the modified Bessel function of the first kind. Under certain conditions, (29) has a limiting Gompertz distribution as $i \rightarrow \infty$ (McNeil, 1970). However, using our method, we can find the distribution function and density of W_i without special knowledge of the functional form (29). Figure 2 shows the exact probability density of W_i for $i = 1, 2, 3, 4, 5$ with $\lambda = 0.1$ and $\mu = 0.5$. The thick black traces are the exact solutions obtained from (29) and the thin red overlaid traces are the solutions computed using the numerical method outlined in this paper. The exact and numerically computed solutions are equal

to high accuracy. The benefit of our numerical method is that it is generic – it works for any BDP with arbitrary rates $\{\lambda_n\}$ and $\{\mu_n\}$, $n = 0, 1, \dots$. In particular, our method does not require analytic derivation of the orthogonal polynomials and corresponding spectral measure required by the framework of (Karlin and McGregor, 1957a,b).

3.2 Queues

Many important models in queuing theory can be written as general BDPs, and the total waiting time (e.g. customer-hours) during a busy period is an important measure of performance in a queue. In these models, customers arrive into a queue (or buffer) as a Poisson process with rate λ , and waiting customers are removed from the queue with per-customer service rate μ . In the $M/M/\infty$ queue, also known as the immigration-death process, there are infinitely many servers, so the arrival and service (birth and death) rates are $\lambda_n = \lambda$ and $\mu_n = n\mu$ for $n > 0$ and $\lambda_0 = \mu_0 = 0$, so $S = \{0\}$. Let $g(x) = x$ and $\lambda_n^* = \lambda/n$, $\lambda_0^* = 0$, and $\mu_n^* = \mu$ for $n > 0$ and $\lambda_0^* = \mu_0^* = 0$. The distribution of the total waiting time W_i is easily found using our computational method.

In the $M/M/1$ queue, also known as the immigration-emigration model, there is only a single server. The rates are $\lambda_n = \lambda$ and $\mu_n = \mu$ with $g(x) = x$ for all x and $S = \{0\}$. Then the modified transition rates are $\lambda_n^* = \lambda/n$ and $\mu_n^* = \mu/n$ for $n > 0$ and $\lambda_0^* = \mu_0^* = 0$. McNeil (1970) gives an expression for the first passage time of the $M/M/\infty$ queue for use in finding the distribution of a Kendall process. Analytic solution of the forward equations appears intractable (Daley, 1969; Daley and Jacobs Jr, 1969), and McNeil (1970) resorts to calculating the mean and variance. However, we can easily compute the entire probability distribution.

Finally, we address the $M/M/c$ queue in which there are exactly c servers. When there are $n < c$ customers in the queue, the service rate is $n\mu$, and when there are c or more customers, the service rate is $c\mu$ for every $n \geq c$. The rates for the queue are therefore $\lambda_n = \lambda$ and $\mu_n = \min\{n, c\}\mu$. Artalejo and Lopez-Herrero (2001) discuss the length of the busy period and derive recursive expressions for its moments. The truncated nature of the death rates makes analytic solution very difficult, but poses no problem for our method. Figure 3 shows the person-time waited during a busy period in the $M/M/\infty$ and $M/M/c$ queues, for $c = 1, 2, 3, 4, 5$, with customer arrival rate $\lambda = 2$, mean service time $\mu = 1$, and $X(0) = 5$ customers in the queue at the beginning of the time of observation. Again, we have the absorbing state $S = 0$ and $g(n) = n$. The solid line is the probability density of total waiting time in the $M/M/\infty$ queue, the dashed line is the $M/M/5$ queue, dotted $M/M/4$, dot-dash $M/M/3$, long dash $M/M/2$, and two-dash $M/M/1$. Note the starkly different dynamics of the waiting time distributions. We suggest that numerically computing densities such as these might be useful to managers in determining the number of servers in an $M/M/c$ queue that achieves a certain desirable waiting time distribution. For example, Green et al (2006) discusses queuing approaches to optimizing staffing levels in a hospital emergency room, and Kaplan (2012) outlines an approach to staffing covert counter-terrorism agencies.

3.3 Extinction of an allele under mutation and selection

The Moran process of population genetics models the number of alleles of a certain type at a biallelic locus in a population of size N (Moran, 1958). Suppose the two alleles are called A_1 and A_2 and that n individuals currently have the A_1 allele. When an individual dies, it is immediately replaced by a new organism whose allele is drawn randomly from the collection of alleles in existence: A_1 with probability n/N and A_2 with probability $(N - n)/N$. Suppose A_1 confers a selective advantage α to carriers, and A_2 confers advantage $\beta < \alpha$. In the Moran process with mutation, the allele chosen by a new individual can mutate from A_1 to A_2 with probability u and vice versa with probability v .

To derive the birth and death rates for the Moran model with mutation and selection, we outline the possible ways of adding or removing an allele of each type to the population. In order for a new A_1 individual to arise ($n \rightarrow n + 1$), one of the $N - n$ A_2 individuals must die. Then either the new individual must draw its allele from one of the n A_1 alleles in existence without mutation, or it must draw one of the $N - n$ A_2 alleles with mutation to A_1 . Therefore, the addition rate is

$$\lambda_n = (N - n) [\alpha n(1 - u) + \beta(N - n)v], \quad (30)$$

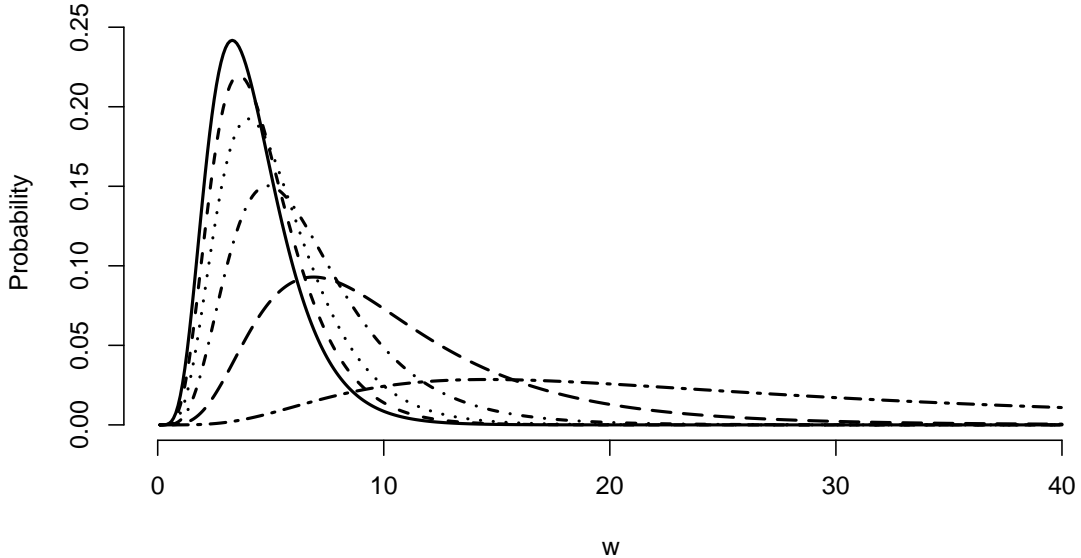


Figure 3: Probability density of the total waiting time in the $M/M/\infty$ and $M/M/c$ queues with arrival rate $\lambda = 2$, mean service time $\mu = 1$, and starting with $X(0) = 7$ customers. The solid trace is the $M/M/\infty$ queue, dashed $M/M/5$, dotted $M/M/4$, dot-dash $M/M/3$, long dash $M/M/2$, and two-dash $M/M/1$. Note the differing shapes of the total waiting time distributions. Numerically computed densities such as these may be useful in determining the optimal number of servers an organization needs in order to achieve a total waiting time distribution with certain desirable properties.

for $n = 0, \dots, N$ with $\lambda_n = 0$ when $n > N$. Likewise, in order for a new A_2 individual to arise ($n \rightarrow n-1$), one of the n A_1 individuals must die. Then either the new individual must draw its allele from one of the $N-n$ A_2 alleles without mutation, or it must draw one of the n A_1 alleles with mutation to A_2 . Therefore, the removal rate is

$$\mu_n = n[\beta(N-n)(1-v) + \alpha nu], \quad (31)$$

for $n = 1, \dots, N$ with $\mu_0 = \lambda_N = 0$ and $\mu_n = 0$ when $n > N$. The Moran model is widely used as a continuous-time alternative to the Wright-Fisher process in population genetics, and expressions for transition probabilities are known in certain simplifying cases (Karlin and McGregor, 1962; Donnelly, 1984). However, the model with selection and mutation is intractable using traditional methods of analysis. However, using our technique, we can compute the full probability distribution of W_i easily and quickly.

Suppose we are interested in the total time lived by organisms carrying the A_1 allele until its eventual extinction, under the assumption that the A_1 allele is disadvantageous ($\alpha < \beta$). Equivalently, we could also follow the total time lived by organisms carrying A_2 until its fixation in the population. Let the total population size be $N = 100$ and suppose $X(0) = 50$. Let the mutation rates be $u = 0$ and $v = 0.03$, so A_1 cannot mutate to A_2 , but the reverse mutation is possible; note that this makes the state 0 absorbing. We therefore let $S = \{0\}$ and observe that τ_{50} , the time to fixation of A_2 , is almost surely finite. Let $\alpha = 0.3$ and $\beta = 0.5$ and $g(n) = n$ for all n . We form the modified process with $\lambda_n^* = \lambda_n/n$ and $\mu_n^* = \mu_n/n$. Figure 4 shows the distribution of the total time lived with the A_1 allele until its eventual extinction in the population.

Integrals of population genetic processes may have uses in conservation and infectious disease settings. If carriers of a certain allele consume a resource (e.g. food) at a different rate from non-carriers, the total time lived by carriers of that allele is proportional to the resource consumed (McNeil, 1970). Alternatively, suppose the differential alleles are carried by a bacterial pathogen that has infected a host animal. If bacteria carrying one of the alleles more readily infects other hosts, then the total time lived by carriers of that allele is related to the likelihood of transmission.

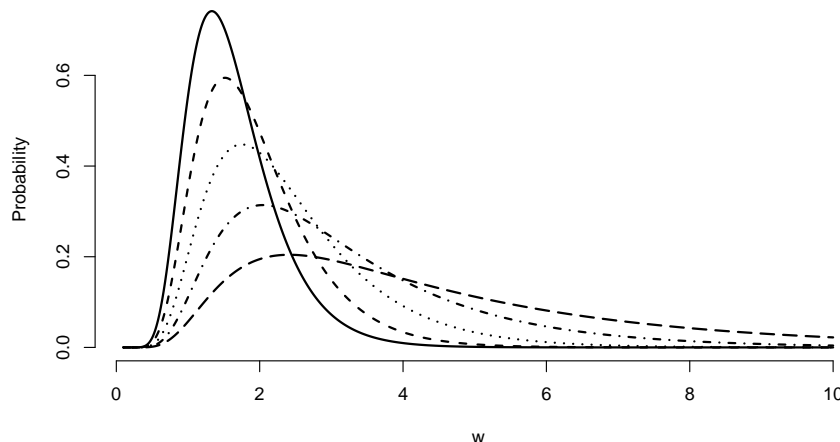


Figure 4: Total time lived by organisms carrying the A_1 allele until its extinction under a Moran model with mutation and selection for different mutation probabilities. The fitness of allele A_1 is $\alpha = 0.5$, the fitness of allele A_2 is $\beta = 1$, there are $N = 50$ individuals in the population, and there are $X(0) = 25$ with the A_1 allele at the start of observation. The traces show the probability density of organism-time lived with the A_1 allele until its eventual extinction, for different probabilities of mutation from A_1 to A_2 is v , with $v = 0.01$ (solid line), $v = 0.02$ (dashed), $v = 0.03$ (dotted), $v = 0.04$ (dash-dotted), $v = 0.05$ (long dash). Note that even very small changes in the mutation dramatically alter the total lifetime distribution.

3.4 Perpetual integral options

The standard stochastic models for the evolution of asset prices generally have continuous sample paths (e.g. Weiner processes). However, there is increasing interest in discrete-valued processes in a variety of contexts. Perhaps the most important is the increasing availability of near-real-time financial data (Engle, 2000). For example, (Barndorff-Nielsen et al, 2010) study low-latency trading data using discrete-valued Lévy processes. Darolles et al (2000) use a BDP to model intra-day transaction prices. Kou and Kou (2003) model the market capitalization of “growth stocks” as a linear BDP with immigration and emigration and derive a variety of useful quantities related to the equilibrium and transient characteristics of the process. In this section, we show how a certain type of financial option can be analyzed using our framework for studying integral functionals of general BDPs.

An *option* is a contract to buy or sell a specified asset sometime in the future (Hull, 2009). An American option may be exercised at any time before its expiry date, and a perpetual American option can be exercised at any time – it does not expire. A perpetual American integral option exercised at time t entitles the buyer to receive payment proportional to the integral of the price of the asset up to time t (Kramkov and Mordecki, 1994):

$$\int_0^t g(X(t)) dt \quad (32)$$

Usually, the reward function has the form $g(n) = an + bi$ where $a, b > 0$ and $i = X(0)$. The arbitrage-free price of an American integral option is defined as the largest expected value over all stopping rules τ of the reward integral:

$$V_* = \sup_{\tau} \mathbb{E} \left[\int_0^{\tau} g(X(t)) dt \right] \quad (33)$$

Some authors, discussing stochastic differential equation models for asset pricing, have shown that the optimal time to exercise an American perpetual option is a stopping rule for the first passage time to a state k such that the payoff is maximized (Kyprianou and Pistorius, 2003). That is,

$$\tau = \arg \max_k \left\{ \int_0^{\tau_i(k)} g(X(t)) dt : \tau_i(k) = \inf \{t : X(t) = k \mid X(0) = i\} \right\} \quad (34)$$

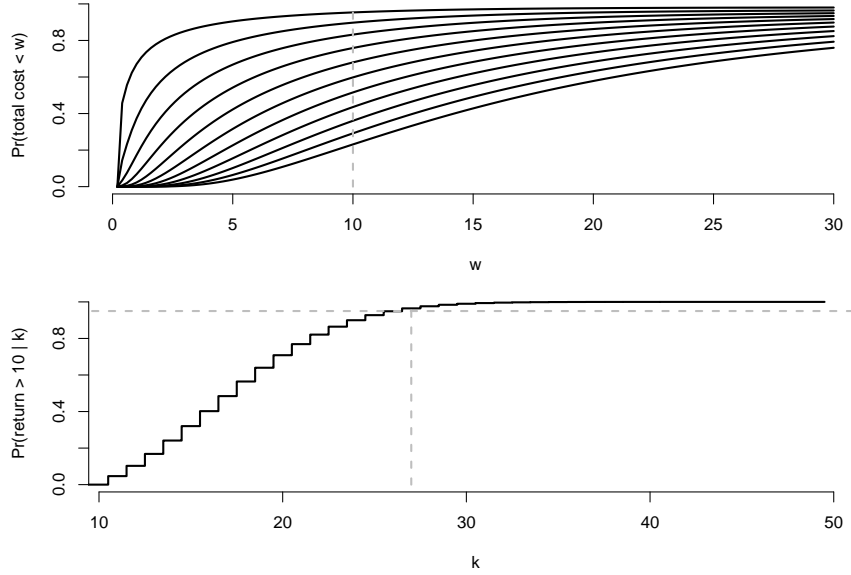


Figure 5: Determination of exercise price in an undiscounted arbitrage-free American perpetual integral option. The top panel shows the distribution of the return on a perpetual integral option, $\Pr(W_i(k) < w)$ for $k = 11, \dots, 21$. The vertical line is at $W_i = 10$, the desired payoff. The bottom panel shows $\Pr(W_i(k) < 10)$ as a function of the price k . The horizontal dashed line shows 95% probability and the vertical dashed line shows that $k = 27$ achieves the desired bound. That is, the exercise price $k = 27$ guarantees that the return will be large enough, with high probability: $\Pr(W_i(27) > 10) > 0.95$.

Our interest here is not the price of the option, which is given by the expectation in (33) and can be computed for a given k using the methods described by Hernández-Suárez and Castillo-Chavez (1999), Stefanov and Wang (2000), and Pollett (2003). Since our numerical method gives us access to the entire distribution of the integral for a given stopping time, we can instead seek the stopping time that guarantees a certain return with high probability. More formally, we wish to find the smallest exercise price $k > X(0)$, and therefore the shortest time $\tau_i(k)$, that gives

$$\Pr(W_i(k) < R) < \alpha \quad (35)$$

for a desired return $R > 0$ and a small probability $0 < \alpha < 1$, where $W_i(k)$ is the integral with the stopping time $\tau_i(k)$.

Suppose $X(t)$ is the price of an asset that moves in unit price increments (ticks) according to a BDP with $\lambda_n = n\lambda + \alpha$ and $\mu_n = n\mu + \beta$ for $n \geq 1$ with $\mu_0 = 0$. Figure 5 shows an application. The top panel shows the cumulative distribution of W_i for $\lambda = 2$, $\mu = 1.5$, $\alpha = 0.3$, and $\beta = 0.5$, with $a = 0$ and $b = 1$ for $k = 11, \dots, 21$ and $X(0) = 10$. The bottom panel shows $\Pr(W_i > 10)$ as a function of k . The stopping price $k = 27$ is the lowest strike price to achieve $\Pr(W_i(k) > 10) > 0.95$. That is, $\tau_i(27)$ is the *shortest* time we must wait to ensure that the return is greater than 10, with greater than 95% probability.

We note that our approach here does not relate directly to the traditional problems of pricing of options or finding the optimal time to exercise. Rather, our method allows selection of an exercise strategy that provides a probabilistic bound on the size of the return. One drawback of our approach is that we are unable to provide the same result for a *discounted* return (e.g. $e^{-r\tau_i}W_i$), which arises when one desires the ratio of return to that of a riskless investment with interest rate r . The difficulty arises because the modified process $e^{rt} \int_0^t X(s) ds$ no longer enjoys the Markov property providing exponential waiting times. We are actively working on a solution for discounted processes.

3.5 Probabilistic control of an epidemic

In infectious disease epidemiology, stochastic modeling can give valuable insight into both disease dynamics and optimal intervention strategies (Wickwire, 1977; Ball, 1986). To illustrate, we model the number of infected persons in a homogeneously mixing population as a type of general BDP. This simple model, called the susceptible-infected-susceptible (SIS) model, keeps track of the number of infected in a finite population of size N Bailey (1957). If there are currently $n < N$ infected persons in the population, the rate of new infections is proportional to the product of the number of infected n and susceptible $N - n$. The contact/transmission rate between infected and susceptible persons is λ . Infected persons recover and revert to susceptible status with constant per-person rate μ . For a SIS process $X(t)$, the addition and removal rates are

$$\lambda_n = \lambda n(N - n) \quad \text{and} \quad \mu_n = n(\mu + \epsilon) \quad (36)$$

where ϵ is a positive control parameter related to vaccination or some other public health intervention strategy. Suppose the initial number of infected is $X(0) = i \leq N$ and we are interested in the total cost of the epidemic until its eventual extinction, so $S = \{0\}$. Let the cost of managing the epidemic per unit time be $a\epsilon$. Additionally, let the cost per infected person per unit time be $b > 0$, so the cost function becomes $g(n) = a\epsilon + bn$. Then the total cost is

$$W_i = \int_0^{\tau_i} [a\epsilon + bX(t)] dt = a\epsilon\tau_i + b \int_0^{\tau_i} X(t) dt \quad (37)$$

where τ_i is the time to extinction of the epidemic.

As in the previous examples in this section, we can easily compute the distribution of total cost W_i . However, this time we go a step further and seek the smallest value of the control parameter that provides a desired probability bound on the total cost of the epidemic. Most optimal control models seek a policy that minimizes the expected total cost, corresponding to the expectation of (1) under certain conditions on the intervention and cost functions (Lefèvre, 1981; Cai and Luo, 1994; Clancy, 1999; Guo and Hernández-Lerma, 2009). But with probability distributions in hand, we can attack the problem in a subtler way. The availability of probability distributions for the total cost of the epidemic allows us to seek the *minimal* intervention policy that guarantees that the total cost of the epidemic is small with high probability. Let $X(t)$ be the process with rates given by (36) for a certain control setting ϵ . Then we wish to find the smallest ϵ such that

$$\Pr(W_i < C) < 1 - \alpha \quad (38)$$

where C is a desired bound on the total cost, and $0 < \alpha < 1$ is a small probability. The left-hand side of (38) is simply the probability that the total cost of an epidemic is less than a certain cost, which is easily computed using our method. Assuming this probability is continuous and increases monotonically with ϵ near $1 - \alpha$, it is straightforward to find the smallest ϵ that satisfies (38). In our experience, these assumptions are satisfied for all reasonable cost functions. Additionally, this model could easily accommodate a cost function $g(n)$ that incorporates the quantities n , $N - n$, and ϵ in a much more complicated way. We emphasize that it is possible that no ϵ exists to satisfy (38) for certain choices of the total cost bound C and probability α .

Figure 6 shows how to find the minimal ϵ for a SIS process with $N = 100$ individuals, $X(0) = 50$, infectivity $\lambda = 0.1$, recovery rate $\mu = 8$, control cost $a = 0.1$, and per-infected cost $b = 0.3$ per unit time. The top traces show the cumulative distribution function of the total cost for $\epsilon = 0, 0.5, 1, 1.5, 2$. The vertical gray line shows $W_i = 7$, and we wish to keep the total cost less than 7 with probability $1 - \alpha = 0.95$. The bottom trace shows $\Pr(W_i < 7)$ as a function of ϵ . The horizontal gray dashed line shows 0.95 probability, and the vertical gray dashed line shows the smallest value of ϵ that achieves this bound. The result is $\epsilon \approx 3.4$. It is interesting to note that this procedure can be completely automated and does not require analytic insight into the SIS process beyond specification of the rates λ and μ . Most traditional epidemic control methods (i.e. Lefèvre (1981)) seek a policy to control the *mean cost*. The analysis presented here might be useful to public health policymakers whose aim, rather than controlling average cost, is to bound the chance of catastrophe.

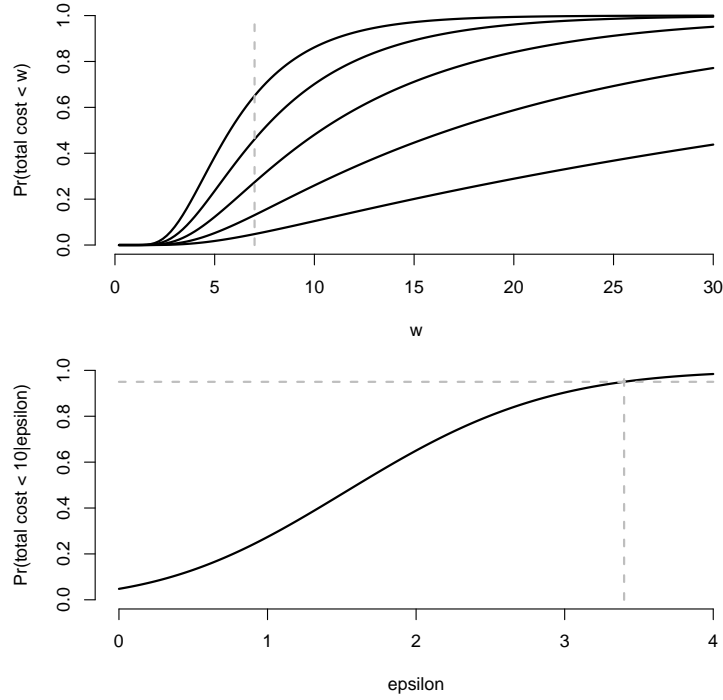


Figure 6: Probabilistic control of a stochastic SIS epidemic. At top, the distribution of total epidemic cost W_i for different values of a control parameter ϵ . The dashed gray vertical line is at $w = 7$, and we wish to keep $W_i < 7$ with high probability. At bottom, the probability that $W_i < 7$ as a function of the control parameter ϵ . The horizontal gray dashed line denotes 0.95, and the vertical dashed line is the smallest epsilon that achieves $\Pr(W_i < 7) > 0.95$; this yields $\epsilon \approx 3.4$. In this way, we can easily find the smallest value of a control parameter that bounds the probability that the epidemic will exceed a certain threshold.

4 Discussion

Development of analytic methods for general BDPs has proceeded down two parallel paths. The first is theoretical and relies on the pioneering work of Karlin and McGregor (1957a,b, 1958) and later developments related to continued fraction expressions for Laplace transformed probabilities by Guillemin and Pinchon (1998, 1999) and Flajolet and Guillemin (2000). This approach benefits from generality – most results apply to all general BDPs, but suffers from a lack of useful time-domain expressions. The second path is more practical and has focused on finding analytic time-domain expressions for probabilities and moments of simple BDPs (see, e.g. Pollett (2003)). Unfortunately, few authors have attempted to bridge this divide, though the work of Murphy and O’Donohoe (1975), Abate and Whitt (1999), and Crawford and Suchard (2012b) are notable exceptions. The main contribution of this work is to reconcile divergent approaches to analyzing integral functionals of general BDPs in the continued fraction framework and to present a unifying representation that is amenable to computational solution.

5 Software

Software implementations of the routines described in this paper are available from the author’s website.

A Appendix: stable evaluation of continued fraction expressions

Evaluating the continued fraction expressions in Theorem 1 can be challenging. The algorithm of Lentz (1976) and subsequent improvements by Thompson and Barnett (1986) and Press (2007) are widely used to evaluate expressions like (7). Crawford and Suchard (2012b) apply this method to compute the Laplace transform of transition probabilities for general BDPs. However, it is also necessary to evaluate ratios of denominators of continued fraction convergents such as

$$\frac{B_{n+1}}{B_n} \quad \text{and} \quad \frac{B_m}{B_n} \quad (39)$$

where $B_n(s)$ is given by the denominator of (10). This appendix describes a stable numerical technique for computing these quantities, based on the Lentz algorithm.

We begin with the following well-known fact about continued fraction convergents.

Lemma 1. *Both the numerator A_k and denominator B_k of (10) satisfy the same recurrence, due to Wallis (1695):*

$$\begin{aligned} A_k &= b_k A_{k-1} + a_k A_{k-2}, \text{ and} \\ B_k &= b_k B_{k-1} + a_k B_{k-2}, \end{aligned} \quad (40)$$

with $A_0 = 0$, $A_1 = a_1$, $B_0 = 1$, and $B_1 = b_1$.

Unfortunately, evaluation of rational approximations to continued fractions using Lemma 1 is subject to substantial roundoff error when the terms become small (Press, 2007). We briefly review the modified Lentz method to compute continued fraction convergents (Lentz, 1976; Thompson and Barnett, 1986; Press, 2007). To approximate the value of $f_{00}(s)$, given by (8), truncating at depth k , we write

$$f_{0,0}^{(k)}(s) = \frac{A_k(s)}{B_k(s)} \quad (41)$$

which is the k th rational approximant to $f_{0,0}(s)$. The modified Lentz method stabilizes the recurrence by instead computing the quantities

$$C_k = \frac{A_k}{A_{k-1}} \quad \text{and} \quad D_k = \frac{B_{k-1}}{B_k} \quad (42)$$

so that

$$f_{0,0}^{(k)} = f_{0,0}^{(k-1)} C_k D_k. \quad (43)$$

and we compute C_k and D_k using Lemma 1 as follows:

$$C_k = b_k + \frac{a_k}{C_{k-1}} \quad \text{and} \quad D_k = \frac{1}{b_k + a_k D_{k-1}}. \quad (44)$$

Returning to the issue of computing ratios of convergents, computing the numerator and denominator directly by (40) is appealing, but prone to catastrophic roundoff error. For concreteness, assume $m \leq n$, and in particular, $n = m + j$ where j is a non-negative integer. Then we must evaluate

$$\frac{B_{n+1}}{B_n} \quad \text{and} \quad \frac{B_m}{B_n} \quad (45)$$

The first of these is simply $1/D_{n+1}$. The second we can find from 1. Let

$$Z_{m+j} = \frac{B_{m+j}}{B_m}. \quad (46)$$

If $j = 0$, then we have

$$Z_m = B_m/B_m = 1 \quad (47)$$

If $j = 1$ then

$$Z_{m+1} = B_{m+1}/B_m = 1/D_m \quad (48)$$

Now using 1, divide the expression for B_{k+2} by B_k to form a new recurrence Z_k as follows:

$$\begin{aligned} Z_{m+2} &= \frac{B_{m+2}}{B_m} \\ &= b_{m+2} \frac{B_{m+1}}{B_m} + a_{m+2} \frac{B_m}{B_m} \\ &= b_{m+2} \frac{1}{D_m} + a_{m+2} \end{aligned} \quad (49)$$

Continuing along these lines, we have

$$Z_{m+3} = b_{m+3} Z_{m+2} + a_{m+3} \frac{1}{D_m} \quad (50)$$

and in general,

$$Z_{m+j} = b_{m+j} Z_{m+j-1} + a_{m+j} Z_{m+j-2}. \quad (51)$$

This is a stable recurrence for computing B_{m+j}/B_m , as desired.

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